Solution of Algebra-II mid semestral exam, M.Math, 2010

February 17, 2016

Question 1: Let F be a finite extension of $\mathbb{Q}$, prove that the algebraic closure of F and $\mathbb{Q}$ are isomorphic.

Solution to question 1

Let F be a finite extension of $\mathbb{Q}$, let α be algebraic over F, then the extension $F(\alpha)$ is finite over F and F is finite over $\mathbb{Q}$. Therefore the extension $F(\alpha)$ over $\mathbb{Q}$ is finite. Therefore α is algebraic over $\mathbb{Q}$. Therefore $\bar{F}$ is contained in $\bar{\mathbb{Q}}$. On the other hand we have $\bar{\mathbb{Q}}$ embedded inside $\bar{F}$. Therefore the inclusion homomorphism from $\bar{\mathbb{Q}}$ to $\bar{F}$ is surjective, hence an isomorphism.

Question 2: Let K be a finite separable extension of F and let G be the group of automorphisms of K over F. Then prove that $\{x \in K | g(x) = x, \forall g \in G\}$ equals F.

Solution to question 2

Let K be a finite separable extension of F. Let G denote the group of automorphisms of K over F. We have to prove that $\{x \in K | g(x) = x\}$ is equal to F. Let α be such that $g(\alpha) = \alpha$ for all g in $Aut(K/F)$. Since K over F is separable, α satisfies a separable polynomial $p(x)$, where $p(x)$ is in $F[x]$. We write $p(x)$ as

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

since

$$p(\alpha) = 0$$

we have

$$a_0 + a_1 \alpha + a_2 \alpha^2 + \cdots + a_n \alpha^n = 0$$

applying $g \in Aut(K/F)$ we get that

$$a_0 + a_1 g(\alpha) + \cdots + a_n g(\alpha)^n = 0$$

this is because $a_i = g(a_i)$, since a_i belongs to K. The above tells us that $g(\alpha)$ is a root of $p(x)$. But $g(\alpha) = \alpha$ and $p(x)$ has distinct roots so we get that $p(x) = x - \alpha$, so α belongs to F.

Question 3: Determine the degree of the splitting field of $x^3 + 2x + 1, x^3 - 5$ over $\mathbb{Q}$, justify the answer.

Solution of problem 3

I) The discriminant of $x^3 + 2x + 1 = 0$ is -59, so one root is real and other two are complex conjugates. We can compute that the real is not a rational number. Therefore the extension attaching the real root is 3. Then the rest of the two complex conjugate roots satisfy a quadratic polynomial. So the degree of the splitting field is 6.

II) The roots are x^5-3 are $\sqrt[5]{3}$ and $\zeta\sqrt[5]{3}$ where ζ is a 5-th root of unity. Let ζ_5 be a primitive 5-th root os unity. Since the ratio of $\zeta_5\sqrt[5]{3}$ and $\sqrt[5]{3}$ is in the splitting field we have ζ_5 is in splitting field. So the splitting field is contains $\mathbb{Q}[\zeta_5, \sqrt[5]{3}]$ and also since the roots are $\sqrt[5]{3}$ and $\zeta\sqrt[5]{3}$, we have the splitting field is exactly $\mathbb{Q}[\zeta_5, \sqrt[5]{3}]$. Now this extension is nothing but $\mathbb{Q}[\zeta_5][\sqrt[5]{3}]$, which is of degree atmost 5 over the cyclotomic field $\mathbb{Q}[\zeta_5]$. So the degree of the extension $\mathbb{Q}[\zeta_5, \sqrt[5]{3}]$ is atmost $5.4 = 20$ (Since $\mathbb{Q}[\zeta_5]$ over $\mathbb{Q}$ is of degree 4). Also $\mathbb{Q}[\sqrt[5]{3}]$ and $\mathbb{Q}[\zeta_5]$ are two subfields of the splitting field, so we have 20 divides the degree of the extension $\mathbb{Q}[\zeta_5, \sqrt[5]{3}]$. Therefore the degree is exactly 20.

Question 4: Let f be an irreducible degree six polynomial over a field F and let K be a degree 2 extension of F. Show that f either remains irreducible over K or splits as a product of two irreducible cubics.

Solution to problem 4

Let K be a degree 2 extension over F. Let f be a degree 6 irreducible polynomial over F. Suppose f splits over K. Attach a root of f to K. Then the extension $K(\alpha)$ over F is divisible by 2, since degree of K over F is 2. On the other hand degree of $K(\alpha)$ over F is 6, since f is of degree 6. Therefore extension $K(\alpha)$ over K is of degree 3. So f breaks up as a product of two cubic polynomials over K. Otherwise it is irreducible.

Question 5: Define the separable degree of a field extension. Prove that the separable degree of a finite extension is bounded by the degree of that extension.

Solution to problem 5

Let K over F is a finite extension. Consider the subset E of all elements in K which are separable over F. Then E is an intermediate field extension of F and lying in K. The degree of the extension E over F is called the separable degree of the extension K over F.

By definition of separable degree it is bounded by the number $[K : F]$.

Question 6: Prove that -1 is not a square in $\mathbb{Q}[\sqrt[4]{-2}]$.

Solution to problem 6

Suppose that -1 is a square in $\mathbb{Q}[\sqrt[4]{-2}]$. Then we have

$$i^2 = -1, \quad i^4 = 1$$

so $i\sqrt[4]{-2}$ is also a root of $\mathbb{Q}[\sqrt[4]{-2}]$. So the splitting field of $x^4 + 2 = 0$ can be written as $\mathbb{Q}[\sqrt[4]{-2}, i]$ and since i belongs to $\mathbb{Q}[\sqrt[4]{-2}]$ we have the degree of the extension $\mathbb{Q}[\sqrt[4]{-2}, i]$ is 4. On the other hand we show that $\sqrt[4]{-2}$ is not in $\mathbb{Q}[i]$. So the degree of the extension $\mathbb{Q}[i, \sqrt[4]{-2}]$ is 8. So we get a contradiction that i does not belong to $\mathbb{Q}[\sqrt[4]{-2}]$. So let

$$a + ib = \sqrt[4]{-2}$$

where a, b are rational numbers. From this we get that

$$a^4 - 6a^2b^2 + b^4 = -2$$

$$4ab(a^2 - b^2) = 0$$

so we have $a = 0$ or $b = 0$ or $a = \pm b$. Therefore we get that $a^4 = -2$ or $b^4 = -2$ or $4a^4 = 2$ all of which are absurd since a, b are rational numbers. So i does not belong to $\mathbb{Q}[\sqrt[4]{-2}]$ and hence -1 is not a square in $\mathbb{Q}[\sqrt[4]{-2}]$.